The spectral linewidth of laser noise and of fluctuations near other second-order phase transitions

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Abstract. A simple, exact analytic expression is derived for the effective spectral linewidth of laser intensity fluctuations as a function of output power.

Relevance of the work to the general theory of second-order phase transitions is pointed out.

The dependence of the spectral linewidth of laser amplitude noise on output intensity has been computed numerically and verified experimentally by several authors (for a review see Risken 1970). In this letter we point out that an experimentally well defined and useful effective linewidth may be derived from the fundamental density-matrix equation for the laser radiation field by analytical techniques. The simple exact formula so obtained gives an output-dependent effective linewidth differing marginally from the previously computed values, since these were defined experimentally in a slightly different manner.

The intensity autocorrelation function, that is, the Fourier transform of the noise spectrum, may be written

$$G^{(2)}(\tau) = \sum_{n,m} nm\, \rho(m, 0; n, \tau) \tag{1}$$

where $\rho(m, 0; n, \tau)$ is the joint probability of finding m photons in the field at time t and n at time $t+\tau$. Equation (1) may be re-interpreted in terms of the diagonal elements ρ_{nn} of the radiation-field density matrix as follows:

$$G^{(2)}(\tau) = \sum_{n,m} nm\, \rho_{mm}(0)\rho_{nn}(m, 0|\tau) \tag{2}$$

where $\rho_{mm}(0)$ is the equilibrium photon distribution and $\rho_{nn}(m, 0|\tau)$ the conditional probability of finding n photons at time $t+\tau$ after finding m at time t. We define an effective intensity fluctuation linewidth λ_{eff} in terms of the slope of $G^{(2)}(\tau)$ at the origin by the relation

$$G^{(2)}(\tau) = \bar{n}^2\{1+(n^{(2)}-1)\exp(-\lambda_{\text{eff}}\tau)\}+\mathrm{O}(\tau^2) \tag{3}$$

so that

$$\lambda_{\text{eff}} = -\left.\frac{\mathrm{d}G^{(2)}(\tau)}{\mathrm{d}\tau}\right|_0 \bigg/ \bar{n}^2(n^{(2)}-1) \tag{4}$$

where $\bar{n}$ is the mean photon number and $n^{(2)}$ the normalized second factorial moment of the photon number distribution. From (2), (3) and (4) we then have

$$\lambda_{\text{eff}} = \frac{1}{\bar{n}^2(n^{(2)}-1)} \sum_{n,m} nm\, \rho_{mm}(0) \left.\frac{\mathrm{d}\rho_{nn}}{\mathrm{d}\tau}(m, 0|\tau)\right|_0. \tag{5}$$

The quantity $\mathrm{d}\rho_{nn}/\mathrm{d}\tau|_0$ may be evaluated from the equation satisfied by the density matrix of the laser radiation (Scully *et al.* 1966, Scully and Lamb 1967):

$$\frac{\mathrm{d}\rho_{nn}(\tau)}{\mathrm{d}\tau} = -\frac{A(n+1)\rho_{nn}}{1+(n+1)B/A} + \frac{An}{1+nB/A}\rho_{n-1,n-1} - Cn\rho_{nn} + C(n+1)\rho_{n+1,n+1}. \tag{6}$$

Bearing in mind its definition, it is clear that $\rho_{nn}(m, 0|0) = \delta_{mn}$, so that $\mathrm{d}\rho_{nn}/\mathrm{d}\tau|_0$

vanishes according to equation (6) unless $n = m$, $m+1$ or $m-1$. Substituting for these three terms in equation (5) and performing the sum over m using the equilibrium distribution (Scully and Lamb 1967)

$$\rho_{mm}(0) \propto \left(\frac{A^2}{BC}\right)^m \Big/ \Gamma\left(1+m+\frac{A}{B}\right) \tag{7}$$

leads finally to the simple formula

$$\lambda_{\text{eff}} = -C/\bar{n}(n^{(2)}-1). \tag{8}$$

In terms of the threshold values $\lambda_{\text{eff}}{}^0$, $\bar{n}_0$ and $n_0{}^{(2)}$:

$$\frac{\lambda_{\text{eff}}}{\lambda_{\text{eff}}{}^0} = \frac{\bar{n}_0(n_0{}^{(2)}-1)}{\bar{n}(n^{(2)}-1)}. \tag{9}$$

These formulae may be evaluated sufficiently accurately by using the approximation —valid for large photon number at threshold—first given by Risken (1965):

$$\frac{\bar{n}}{\bar{n}_0} = \sqrt{(\pi)}w + \frac{\exp(-w^2)}{1+\operatorname{erf}(w)} \tag{10}$$

$$n^{(2)} = \pi\left(\frac{\bar{n}_0}{\bar{n}}\right)^2\left(\frac{w\bar{n}}{\sqrt{(\pi)}\bar{n}_0}+\frac{1}{2}\right) \tag{11}$$

where, in Lamb and Scully's notation,

$$w = \left(\frac{A}{B}\right)^{1/2}\left(\frac{A}{C}-1\right). \tag{12}$$

In the limits of large and small output power we have the asymptotic relation

$$\frac{\lambda_{\text{eff}}}{\lambda_{\text{eff}}{}^0} \to 2|w|\left(\frac{\pi/2-1}{\sqrt{\pi}}\right). \tag{13}$$

The recent identification of laser theory with other second-order phase transition theories (Graham and Haken 1970, Scully and DeGiorgio 1970) will doubtless give our simple formula, equation (8), an interesting wider application. The fact, for instance, that the minimum linewidth does not occur exactly at the critical point (at the minimum $\lambda_{\text{eff}}/\lambda_{\text{eff}}{}^0 = 0{\cdot}930\,433 \pm 0{\cdot}000\,001$, $w = 0.641 \pm 0{\cdot}001$, $\bar{n}/\bar{n}_0 = 1{\cdot}542 \pm 0{\cdot}001$) suggests, perhaps, an oversimplification in modern scaling theories.

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